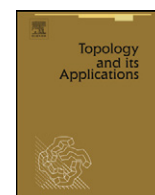


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On functors preserving skeletal maps and skeletally generated compacta

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ABSTRACT

A map $f : X \rightarrow Y$ between topological spaces is *skeletal* if the preimage $f^{-1}(A)$ of each nowhere dense subset $A \subset Y$ is nowhere dense in X . We prove that a normal functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ is skeletal (which means that F preserves skeletal epimorphisms) if and only if for any open surjective map $f : X \rightarrow Y$ between metrizable zero-dimensional compacta with two-element non-degeneracy set $N^f = \{x \in X : |f^{-1}(f(x))| > 1\}$ the map $Ff : FX \rightarrow FY$ is skeletal. This characterization implies that each open normal functor is skeletal. The converse is not true even for normal functors of finite degree. The other main result of the paper says that each normal functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ preserves the class of skeletally generated compacta. This contrasts with the known Ščepin's result saying that a normal functor is open if and only if it preserves the class of openly generated compacta.

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1. Introduction

In this paper we address the problem of preservation of skeletal maps and skeletally generated compacta by normal functors in the category **Comp** of compact Hausdorff spaces and their continuous maps. In the sequel all spaces are Hausdorff and all maps are continuous. A *compactum* is a compact Hausdorff space.

A map $f : X \rightarrow Y$ between topological spaces is called *skeletal* if for each nowhere dense subset $A \subset Y$ the preimage $f^{-1}(A)$ is nowhere dense in X , see [10]. It is easy to see that each open map is skeletal while the converse is not true.

To formulate our main results, we need to recall some definitions from the topological theory of functors, see [12]. A functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ in the category **Comp** is defined to be

- *monomorphic* (resp. *epimorphic*) if for each injective (resp. surjective) map $f : X \rightarrow Y$ between compacta the map $Ff : FX \rightarrow FY$ is injective (resp. surjective);
- *(finitely) open* if for any open surjection $f : X \rightarrow Y$ between (finite) compacta the map $Ff : FX \rightarrow FY$ is open;
- *(finitely) skeletal* if for any skeletal surjection $f : X \rightarrow Y$ between (finite) compacta the map $Ff : FX \rightarrow FY$ is skeletal;
- *weight-preserving* if $w(FX) \leq w(X)$ for each infinite compactum X .

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Here $w(X)$ denotes the *weight* (i.e. the smallest cardinality of a base of the topology) of X . If a functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ is monomorphic, then for each closed subspace X of a compact space Y the map $Fi : FX \rightarrow FY$ induced by the inclusion $i : X \rightarrow Y$ is a topological embedding, which allows us to identify the space FX with the subspace $Fi(FX)$ of FY .

Next we recall and define several properties of functors related to the bicommutativity. Let \mathcal{D} be a commutative square diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ p_X \downarrow & & \downarrow p_Y \\ X & \xrightarrow{f} & Y \end{array}$$

consisting of continuous maps between compact spaces.

The diagram \mathcal{D} is called *bicommutative* if $\tilde{f}(p_X^{-1}(x)) = p_Y^{-1}(f(x))$ for all $x \in X$, see [9, §3.IV], [11, §2.1].

We say that a functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$

- is *(finitely) bicommutative* if F preserves the bicommutativity of square diagrams \mathcal{D} consisting of surjective maps f, \tilde{f} , p_X, p_Y and (finite) compacta $X, Y, \tilde{X}, \tilde{Y}$;
- *preserves (finite) preimages* if F preserves the bicommutativity of square diagrams \mathcal{D} with injective maps f, \tilde{f} (and finite space X);
- *preserves (finite) 1-preimages* if F preserves the bicommutativity of square diagrams \mathcal{D} with injective maps f, \tilde{f} , bijective map p_X (and finite space X).

It is clear that each bicommutative functor is finitely bicommutative. The converse is true for normal functors with finite supports, see Proposition 2.10.1 of [12]. It is easy to see that a monomorphic functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ preserves [finite] (1-)preimages if and only if for any map $f : X \rightarrow Y$ between compact spaces and a [finite] closed subset $Z \subset Y$ (such that $f^{-1}(z)$ is a singleton for every $z \in Z$) we get $(Ff)^{-1}(FZ) = F(f^{-1}(Z))$.

A functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ will be called *mec* if F is monomorphic, epimorphic, and continuous. A *mec* functor that preserves finite 1-preimages will be called a *1-mec* functor. A 1-mec functor that preserves weight of infinite compacta will be called a *1-mecw* functor. The class of 1-mecw functors includes all normal functors in the sense of Ščepin [11], [12, §2.3] (let us recall that a functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ is *weakly normal* if it is monomorphic, epimorphic, continuous and preserves intersections, the empty set, the singleton, and the weight of infinite compacta; F is *normal* if it is weakly normal and preserves preimages).

Our primary aim is to characterize skeletal functors among 1-mec functors. For a topological space Z consider the open map $2_Z : Z \oplus 2 \rightarrow Z \oplus 1$ defined by

$$2_Z : z \mapsto \begin{cases} z & \text{if } z \in Z, \\ 0 & \text{if } z \in 2. \end{cases}$$

Here for a natural number n by $Z \oplus n$ we denote the topological sum of Z and the discrete space $n = \{0, \dots, n-1\}$.

Theorem 1.1. A 1-mec (resp. 1-mecw) functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ is skeletal if and only if for each zero-dimensional compact (metrizable) space Z the map $F2_Z : F(Z \oplus 2) \rightarrow F(Z \oplus 1)$ is skeletal.

Since each open map is skeletal, the preceding theorem implies:

Corollary 1.2. Each open 1-mec functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ is skeletal.

Examples 12.3–12.5 presented in Section 12 show that Corollary 1.2 cannot be reversed.

Next, we discuss the interplay between the skeletality and the (finite) bicommutativity of functors.

Theorem 1.3. A 1-mecw functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ is skeletal if it is finitely bicommutative and finitely skeletal.

This criterion should be compared with the following characterization of open functors due to Ščepin, see Propositions 3.18 and 3.19 of [11].

Theorem 1.4 (Ščepin). A normal functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ is open if and only if F is bicommutative and finitely open.

Now let us discuss the problem of preservation of skeletally generated compacta by normal functors. Following [13] we say that a compact Hausdorff space X is *skeletally generated* if X is homeomorphic to the limit of an inverse continuous ω -spectrum $S = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ consisting of metrizable compacta and surjective skeletal bonding projections $\pi_\alpha^\beta : X_\beta \rightarrow X_\alpha$.

According to [8] or [13], a compact Hausdorff space X is skeletally generated if and only if the first player has a winning strategy in the following open–open game. The player I starts the game selecting a non-empty open set V_0 and the player II responds with a non-empty open set $W_0 \subset V_0$. At the n -th inning the player I chooses a non-empty open set V_n and player II responds with a non-empty open set $W_n \subset V_n$. At the end of the game the player I is declared the winner if the union $\bigcup_{n \in \omega} W_n$ is dense in X . Otherwise the player II wins the game.

The class of skeletally generated compacta contains all openly generated compacta and all continuous images of openly generated compacta, see [13]. In particular, each dyadic compactum is skeletally generated. Skeletally generated compacta share some properties of dyadic compacta. In particular, each skeletally generated compactum has countable cellularity, see [4] or [8].

It is known that in general, normal functors do not preserve openly generated compacta. In fact, a normal functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ preserves the class of openly generated spaces if and only if F is open, see [11, §4.1]. This contrasts with the following theorem.

Theorem 1.5. *Each 1-mecw functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ preserves the class of skeletally generated compacta.*

For preimage preserving mecw functors F with $F1 \neq F2$ this theorem can be improved as follows. Below we identify a natural number n with the discrete space $n = \{0, \dots, n-1\}$.

Theorem 1.6. *Let $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ be a preimage preserving mecw functor with $F1 \neq F2$. A compact Hausdorff space X is skeletally generated if and only if the space FX is skeletally generated.*

Remark 1.7. Among the properties composing the definition of 1-mec functor the least studied is the property of preservation of finite 1-preimages. It is clear that a functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ preserves (finite) 1-preimages if it preserves (finite) preimages. On the other hand, the functor of superextension λ and the functor of order-preserving functionals O preserve 1-preimages but fail to preserve finite preimages, see [7], [12, 2.3.2, 2.3.6]. A simple example of a mec functor that does not preserve finite 1-preimages is Pr^3 , the functor of the third projective power, see [12, 2.5.3]. Another example of such a functor is E , the functor of non-expanding functionals, see [7]. By Theorem 1 of [7], a continuous monomorphic functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ preserves 1-preimages if and only if its Chigodiz extension $F_\beta : \mathbf{Tych} \rightarrow \mathbf{Tych}$ preserves embeddings of Tychonoff spaces.

Theorems 1.1, 1.3, 1.5 and 1.6 will be proved in Sections 8–11 after some preliminary work done in Sections 2–6. Several examples of skeletal and non-skeletal functors will be given in Section 12. In that section we also pose some open problems related to skeletal functors.

2. Skeletal maps and skeletal squares

In this section we recall the necessary information on skeletal maps between compact spaces. First we introduce the necessary definitions.

Definition 2.1. A map $f : X \rightarrow Y$ between two topological spaces is defined to be

- *skeletal at a point* $x \in X$ if for each neighborhood $U \subset X$ of x the closure $\text{cl}_Y(f(U))$ of $f(U)$ has non-empty interior in Y ;
- *skeletal at a subset* $A \subset X$ if f is skeletal at each point $x \in A$;
- *open at a point* $x \in X$ if for each neighborhood $U \subset X$ of x the image $f(U)$ is a neighborhood of $f(x)$;
- *open at a subset* $A \subset X$ if f is open at each point $x \in A$;
- *densely open* if f is open at some dense subset $A \subset X$.

It is easy to see that each densely open map is skeletal. The converse is true for skeletal maps between metrizable compacta:

Theorem 2.2. *A map $f : X \rightarrow Y$ between compact metrizable spaces is skeletal if and only if it is open at a dense G_δ -subset of X .*

This theorem has been proved in [1]. The metrizability of X is essential in this theorem as shown by the projection $\text{pr} : A \rightarrow [0, 1]$ of the Aleksandrov “two arrows” space A onto the interval, see [5, 3.10.C]. This projection is skeletal but is open at no point of A .

A characterization of skeletal maps between non-metrizable compact spaces was given in [1] in terms of morphisms of inverse ω -spectra with skeletal limit or bonding squares.

Definition 2.3. Let \mathcal{D} denote a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ p_X \downarrow & & \downarrow p_Y \\ X & \xrightarrow{f} & Y \end{array}$$

consisting of maps between compact spaces. The square \mathcal{D} is defined to be

- *open at a point* $x \in X$ if for each neighborhood $U \subset X$ of x the point $f(x)$ has a neighborhood $V \subset Y$ such that $V \subset f(U)$ and $p_Y^{-1}(V) \subset \tilde{f}(p_X^{-1}(U))$;
- *open at a subset* $A \subset X$ if \mathcal{D} is open at each point $x \in A$;
- *densely open* if it is open at some dense subset $A \subset X$;
- *skeletal at a point* $x \in X$ if for each neighborhood $U \subset X$ of x there is a non-empty open set $V \subset Y$ such that $V \subset f(U)$ and $p_Y^{-1}(V) \subset \tilde{f}(p_X^{-1}(U))$;
- *skeletal at a subset* $A \subset X$ if \mathcal{D} is skeletal at each point $x \in A$;
- *skeletal* if \mathcal{D} is skeletal at X .

Remark 2.4. Let A be a subset of the space X in the diagram \mathcal{D} .

- (1) If the diagram \mathcal{D} is skeletal (resp. open) at A , then the map f is skeletal (resp. open) at A .
- (2) The diagram \mathcal{D} is skeletal (resp. open) at A if it is bicommutative and the map f is skeletal (resp. open) at A .
- (3) If the diagram \mathcal{D} is open at A , then it is *bicommutative at A* in the sense that $\tilde{f}(p_X^{-1}(x)) = p_Y^{-1}(f(x))$ for all points $x \in A$.

Remark 2.5. A map $f : X \rightarrow Y$ is skeletal (resp. open) at a subset $A \subset X$ if and only if the square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \text{id}_X \downarrow & & \downarrow \text{id}_Y \\ X & \xrightarrow{f} & Y \end{array}$$

is skeletal (resp. open) at the subset A .

It is easy to see that each densely open square is skeletal. The converse is true in the metrizable case. The following lemma proved in [1] is a “square” counterpart of the characterization Theorem 2.2.

Lemma 2.6. *If in the diagram \mathcal{D} the space X is metrizable and the map p_Y is surjective, then the square \mathcal{D} is skeletal if and only if \mathcal{D} is open at a dense G_δ -subset of X .*

In order to formulate the spectral characterization of skeletal maps, we need to recall some information about inverse spectra from [5, §2.5], [3, Ch. 1] or [6, §3.1].

For an inverse spectrum $S = \{X_\alpha, p_\alpha^\beta, \Sigma\}$ consisting of topological spaces and continuous bonding maps, by

$$\lim S = \left\{ (x_\alpha)_{\alpha \in \Sigma} \in \prod_{\alpha \in \Sigma} X_\alpha : \forall \alpha \leq \beta, p_\alpha^\beta(x_\beta) = x_\alpha \right\}$$

we denote the limit of S and by $p_\alpha : \lim S \rightarrow X_\alpha$, $p_\alpha : x \mapsto x_\alpha$, the limit projections.

An inverse spectrum $S = \{X_\alpha, p_\alpha^\beta, \Sigma\}$ is called an ω -spectrum if

- each space X_α , $\alpha \in \Sigma$, has countable weight;
- the index set Σ is ω -complete in the sense that each countable subset $\Sigma' \subset \Sigma$ has the smallest upper bound $\sup \Sigma'$ in Σ ;
- the spectrum S is ω -continuous in the sense that for any countable directed subset $\Sigma' \subset \Sigma$ with $\gamma = \sup \Sigma'$ the limit map $\lim p_\alpha^\gamma : X_\gamma \rightarrow \lim\{X_\alpha, p_\alpha^\beta, \Sigma'\}$ is a homeomorphism.

Let $S_X = \{X_\alpha, p_\alpha^\beta, \Sigma\}$ and $S_Y = \{Y_\alpha, \pi_\alpha^\beta, \Sigma\}$ be two inverse spectra indexed by the same directed partially ordered set Σ . A *morphism* $\{f_\alpha\}_{\alpha \in \Sigma} : S_X \rightarrow S_Y$ between these spectra is a family of maps $\{f_\alpha : X_\alpha \rightarrow Y_\alpha\}_{\alpha \in \Sigma}$ such that $f_\alpha \circ p_\alpha^\beta = \pi_\alpha^\beta \circ f_\beta$ for any elements $\alpha \leq \beta$ in Σ .

Each morphism $\{f_\alpha\}_{\alpha \in \Sigma} : S_X \rightarrow S_Y$ between inverse spectra induces the limit map

$$\lim f_\alpha : \lim S_X \rightarrow \lim S_Y, \quad \lim f_\alpha : (x_\alpha)_{\alpha \in \Sigma} \mapsto (f_\alpha(x_\alpha))_{\alpha \in \Sigma}$$

between the limit spaces of these spectra.

Following [1] we say that a morphism $\{f_\alpha\}_{\alpha \in \Sigma} : S_X \rightarrow S_Y$ between two inverse spectra $S_X = \{X_\alpha, p_\alpha^\beta, \Sigma\}$ and $S_Y = \{Y_\alpha, \pi_\alpha^\beta, \Sigma\}$:

- is *skeletal* if each map $f_\alpha : X_\alpha \rightarrow Y_\alpha$, $\alpha \in \Sigma$, is skeletal;
- has *skeletal limit squares* if for every $\alpha \in \Sigma$ the commutative square

$$\begin{array}{ccc} \lim S_X & \xrightarrow{\lim f_\alpha} & \lim S_Y \\ p_\alpha \downarrow & & \downarrow \pi_\alpha \\ X_\alpha & \xrightarrow{f_\alpha} & Y_\alpha \end{array}$$

is skeletal.

We say that two maps $f : X \rightarrow Y$ and $f' : X' \rightarrow Y'$ are *homeomorphic* if there are homeomorphisms $h_X : X \rightarrow X'$ and $h_Y : Y \rightarrow Y'$ such that $f' \circ h_X = h_Y \circ f$.

The following spectral characterization of skeletal maps was proved in [1].

Theorem 2.7. For a map $f : X \rightarrow Y$ between compact Hausdorff spaces the following conditions are equivalent:

- (1) f is skeletal.
- (2) f is homeomorphic to the limit map $\lim f_\alpha : \lim S_X \rightarrow \lim S_Y$ of a skeletal morphism $\{f_\alpha\} : S_X \rightarrow S_Y$ between two ω -spectra $S_X = \{X_\alpha, p_\alpha^\beta, \Sigma\}$ and $S_Y = \{Y_\alpha, \pi_\alpha^\beta, \Sigma\}$ with surjective limit projections.
- (3) f is homeomorphic to the limit map $\lim f_\alpha : \lim S_X \rightarrow \lim S_Y$ of a morphism $\{f_\alpha\} : S_X \rightarrow S_Y$ with skeletal limit squares between two ω -spectra $S_X = \{X_\alpha, p_\alpha^\beta, \Sigma\}$ and $S_Y = \{Y_\alpha, \pi_\alpha^\beta, \Sigma\}$ with surjective limit projections.

3. Some properties of densely open squares

In this section we assume that \mathcal{D} is a commutative square

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ p_X \downarrow & & \downarrow p_Y \\ X & \xrightarrow{f} & Y \end{array}$$

consisting of surjective maps between compact spaces.

By

$$D_f = \{y \in Y : |f^{-1}(y)| = 1\} \quad \text{and} \quad D^f = \{x \in X : |f^{-1}(f(x))| = 1\}$$

we denote the *lower* and *upper degeneracy sets* of the map $f : X \rightarrow Y$, respectively.

Lemma 3.1. The square \mathcal{D} is open at the upper degeneracy set $D^f \subset X$ of f .

Proof. Given an open neighborhood $U \subset X$ of a point $x \in D^f$, observe that the set $V = Y \setminus f(X \setminus U)$ is an open neighborhood of $f(x)$ such that $f^{-1}(V) \subset U$. Applying to this inclusion the surjective map f , we get $V \subset f(U)$. To see that $p_Y^{-1}(V) \subset \tilde{f}(p_X^{-1}(U))$, fix any point $\tilde{y} \in p_Y^{-1}(V)$ and using the surjectivity of the map \tilde{f} , find a point $\tilde{x} \in \tilde{X}$ with $\tilde{f}(\tilde{x}) = \tilde{y}$. It follows that $f \circ p_X(\tilde{x}) = p_Y \circ \tilde{f}(\tilde{x}) \in V$ and hence $p_X(\tilde{x}) \in f^{-1}(V) \subset U$. Then $\tilde{x} \in p_X^{-1}(U)$ and $\tilde{y} = \tilde{f}(\tilde{x}) \in \tilde{f}(p_X^{-1}(U))$. \square

Lemma 3.2. Assume that the square \mathcal{D} is open at a point $a \in X$ and the space X is first countable at a . Then there is a closed subset $Z \subset X$ such that $a \in D^{f|Z}$, $f(Z) = Y$ and $\tilde{f}(p_X^{-1}(Z)) = \tilde{Y}$.

Proof. Being first countable at a , the space X has a countable neighborhood base $(W_n)_{n \in \omega}$ at a such that $W_{n+1} \subset W_n \subset W_0 = X$ for all $n \in \omega$. Let $U_0 = X$ and $V_0 = Y$. Using the fact that the square \mathcal{D} is open at the point a , by induction on n , we can construct a sequence $(U_n)_{n=1}^\infty$ of open neighborhoods of a in X and a sequence $(V_n)_{n=1}^\infty$ of open neighborhoods of $f(a)$ in Y such that

- $U_n \subset W_n \cap U_{n-1} \cap f^{-1}(V_{n-1})$,
- $V_n \subset f(U_n) \cap V_{n-1}$ and $p_Y^{-1}(V_n) \subset \tilde{f}(p_X^{-1}(U_n))$

for every $n \in \mathbb{N}$.

We claim that the set

$$Z = \{a\} \cup \bigcup_{n \in \omega} \bar{U}_n \setminus f^{-1}(V_{n+1})$$

is closed in X and has the required properties: $a \in D^{f|Z}$, $f(Z) = Y$ and $\tilde{f}(p_X^{-1}(Z)) = \tilde{Y}$.

The definition of the set Z implies that it is closed in X and $a \in D^{f|Z}$. To show that $\tilde{f}(p_X^{-1}(Z)) = \tilde{Y}$, fix any point $\tilde{y} \in \tilde{Y}$. We need to find a point $\tilde{x} \in p_X^{-1}(Z)$ such that $\tilde{f}(\tilde{x}) = \tilde{y}$.

For this we consider separately two cases.

- (1) The image $y = p_Y(\tilde{y})$ of \tilde{y} coincides with $f(a)$. In this case for every $n \in \omega$ we get $\tilde{y} \in p_Y^{-1}(V_n) \subset \tilde{f}(p_X^{-1}(U_n))$ and hence there is a point $\tilde{x}_n \in p_X^{-1}(U_n)$ such that $\tilde{y} = \tilde{f}(\tilde{x}_n)$. By the compactness of \tilde{X} , the sequence $(\tilde{x}_n)_{n \in \omega}$ has an accumulation point $\tilde{x} \in p_X^{-1}(a) \subset p_X^{-1}(Z)$. The continuity of the map \tilde{f} guarantees that $\tilde{f}(\tilde{x}) = \tilde{y}$.
- (2) The point $y = p_Y(\tilde{y})$ is not equal to $f(a)$. Since $V_0 = Y$ and $\bigcap_{n \in \omega} f(U_n) = \bigcap_{n \in \omega} V_n = \{f(a)\}$, there is a unique number $n \in \omega$ such that $y \in V_n \setminus V_{n+1}$. Then $\tilde{y} \in p_Y^{-1}(V_n) \subset \tilde{f}(p_X^{-1}(U_n))$ and hence there is a point $\tilde{x} \in p_X^{-1}(U_n)$ such that $\tilde{f}(\tilde{x}) = \tilde{y}$. Consider the image $x = p_X(\tilde{x}) \in U_n$ and observe that $f(x) = f \circ p_X(\tilde{x}) = p_Y \circ \tilde{f}(\tilde{x}) = p_Y(\tilde{y}) = y \notin V_{n+1}$. Consequently, $x \in U_n \setminus f^{-1}(V_{n+1}) \subset Z$ and $\tilde{x} \in p_X^{-1}(x) \subset p_X^{-1}(Z)$.

Therefore $\tilde{f}(p_X^{-1}(Z)) = \tilde{Y}$. Applying to this equality the surjective map p_Y , we get

$$f(Z) = f \circ p_X(p_X^{-1}(Z)) = p_Y \circ \tilde{f}(p_X^{-1}(Z)) = p_Y(\tilde{Y}) = Y. \quad \square$$

Lemma 3.3. *If the square \mathcal{D} is open at a finite subset $A \subset X$, the space X is first countable at each point $x \in A$, and the restriction $f|_A$ is injective, then there is a closed subset $Z \subset X$ such that $f(Z) = Y$, $\tilde{f}(p_X^{-1}(Z)) = \tilde{Y}$, and $A \subset D^{f|Z}$.*

Proof. By Lemma 3.2, for every point $a \in A$ there is a closed subset $Z_a \subset X$ such that $a \in D^{f|Z_a}$, $f(Z_a) = Y$, and $\tilde{f}(p_X^{-1}(Z_a)) = \tilde{Y}$. Since $f|_A$ is injective, for each point $a \in A$ we can find an open neighborhood $W_a \subset Y$ of $f(a)$ in Y such that the closures \bar{W}_a , $a \in A$, are pairwise distinct. Let $B = Y \setminus \bigcup_{a \in A} W_a$ and

$$Z = f^{-1}(B) \cup \bigcup_{a \in A} f^{-1}(\bar{W}_a) \cap Z_a.$$

It is easy to check that the set Z is closed and has the required properties: $A \subset D^{f|Z}$, $f(Z) = Y$ and $\tilde{f}(p_X^{-1}(Z)) = \tilde{Y}$. \square

The proof of the following simple lemma is left to the reader.

Lemma 3.4. *A map $f : X \rightarrow Y$ is open (resp. skeletal) at a point $x \in X$ provided for some subset $Z \subset X$ that contains the point x the map $f|_Z : Z \rightarrow Y$ is open (resp. skeletal) at the point x .*

The following lemma is a “square” counterpart of Lemma 3.4.

Lemma 3.5. *The square \mathcal{D} is open (resp. skeletal) at a point $x \in X$ provided for some subset $Z \subset X$ that contains the point x , and its preimage $\tilde{Z} = p_X^{-1}(Z)$ the square*

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{\tilde{f}|_{\tilde{Z}}} & \tilde{Y} \\ p_X|_{\tilde{Z}} \downarrow & & \downarrow p_Y \\ Z & \xrightarrow{f|_Z} & Y \end{array}$$

is open (resp. skeletal) at the point x .

4. Preliminaries on functors

In this section we prove some auxiliary results on functors in the category **Comp** of compacta. From now on we assume that $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ is a monomorphic epimorphic continuous functor.

For two compact Hausdorff spaces X and Y by $C(X, Y)$ we denote the space of continuous functions $f : X \rightarrow Y$, endowed with the compact-open topology. A proof of the following fact due to Ščepin [11, §3.2] can be found in [12, 2.2.3].

Lemma 4.1. *For any compacta X, Y the map*

$$F : C(X, Y) \rightarrow C(FX, FY), \quad F : f \mapsto Ff,$$

is continuous.

Next, we discuss the notion of support. Let X be a compact Hausdorff space, and $a \in FX$. We say that a point $a \in FX$ has *finite support* if $a \in FA$ for some finite subspace $A \subset X$. In this case we define $\text{supp}(a)$ as the intersection

$$\text{supp}(a) = \bigcap \{A : a \in FA, A \subset X \text{ is finite}\}.$$

We shall often use the following fact proved in [2]:

Lemma 4.2. *Let $a \in FX$ be an element with finite support. If $\text{supp}(a) \neq \emptyset$, then $a \in F(\text{supp}(a))$. If $\text{supp}(a) = \emptyset$, then $a \in FA$ for any non-empty closed subspace $A \subset X$.*

The set of all elements with finite support in FX will be denoted by $F_\omega(X)$. The following lemma was proved in [12, 2.2.1].

Lemma 4.3. *The subset $F_\omega(X)$ is dense in FX .*

For a topological space Y by \dot{Y} we shall denote the set of isolated points of Y . For a surjective function $f : X \rightarrow Y$ let

$$N_f = \{y \in Y : |f^{-1}(y)| > 1\} = Y \setminus D_f \quad \text{and} \quad N^f = \{x \in X : |f^{-1}(f(x))| > 1\} = X \setminus D^f$$

be the *lower and upper non-degeneracy sets* of f , respectively.

Lemma 4.4. *For any skeletal map $f : X \rightarrow Y$ between compacta and any dense subset $A \subset X$, the set*

$$\mathcal{A}_0 = \{a \in F_\omega(X) : \text{supp}(a) \subset A, N_{f|_{\text{supp}(a)}} \subset \dot{Y}\}$$

is dense in FX .

Proof. By Lemma 4.3, the set $F_\omega(X)$ is dense in FX . So, it suffices to check that \mathcal{A}_0 is dense in $F_\omega(X)$. Fix any element $a \in F_\omega(X)$ and a neighborhood $O_a \subset F_\omega(X)$. We need to find an element $b \in O_a \cap \mathcal{A}_0$. If $\text{supp}(a) = \emptyset$, then $a \in \mathcal{A}_0 \cap O_a$ by the definition of \mathcal{A}_0 . So we assume that $B = \text{supp}(a)$ is not empty. By Lemma 4.2, $a \in FB$. By Lemma 4.1, the map

$$F : C(B, X) \rightarrow C(FB, FX), \quad F : g \mapsto Fg$$

is continuous and so is the map

$$F_a : C(B, X) \rightarrow FX, \quad F_a : g \mapsto Fg(a).$$

It follows from the continuity of F_a that the identity inclusion $i_B : B \rightarrow X$ has a neighborhood $O(i_B)$ in the function space $C(B, X)$ such that $F_a(g) = Fg(a) \in O_a$ for any map $g \in O(i_B)$.

We claim that there is a map $g \in O(i_B)$ such that $N_{f \circ g|_B} \subset \dot{Y}$. Since the compact-open topology on $C(B, X)$ coincides with the topology of pointwise convergence, for each point $x \in B$ we can find a neighborhood $O_x \subset X$ such that a map $g : B \rightarrow X$ belongs to the neighborhood $O(i_B)$ provided $g(x) \in O_x$ for all $x \in B$. Let $C = B \cap f^{-1}(\dot{Y})$.

We claim that for each point $x \in B \setminus C$ the set $f(O_x)$ is infinite. Assuming the converse, we can find a smaller neighborhood U_x of x such that $f(U_x)$ coincides with the singleton $\{f(x)\}$ which is open in Y because of the skeletal property of f . In this case $f(x) \in \dot{Y}$ and $x \in C$, which contradicts the choice of x .

Let $B \setminus C = \{x_1, \dots, x_n\}$ be an enumeration of the set $B \setminus C$. By finite induction for every $i \leq n$ choose a point $x'_i \in O_{x_i} \cap A$ such that $f(x'_i) \notin f(C) \cup \{f(x'_j) : j < i\}$. As $f(O_{x_i})$ is infinite and A is dense in X , the choice of x'_i is always possible. After completing the inductive construction, define a map $g : B \rightarrow X$ letting $g(x_i) = x'_i$ for $i \leq n$ and $g(x) \in O_x \cap A \cap f^{-1}(f(x))$ for any $x \in C$. By the construction, $g \in O(i_B)$ and the map $f \circ g|_B \setminus C$ is injective, which means that $N_{f \circ g|_B} \subset f(C) \subset \dot{Y}$.

By the choice of the neighborhood $O(i_B)$, the element $b = Fg(a)$ lies in the neighborhood O_a . Since $b \in F(g(B))$, we get $\text{supp}(b) \subset g(B) \subset A$, witnessing that $b \in \mathcal{A}_0$. \square

Lemma 4.5. *Let $f : X \rightarrow Y$ be a skeletal map between compact Hausdorff spaces. If $\dot{Y} \subset D_f$, then for every dense subset $A \subset X$ the set*

$$\mathcal{A}_1 = \{a \in F_\omega(X) : \text{supp}(a) \subset A, f|_{\text{supp}(a)} \text{ is 1-to-1}\}$$

is dense in FX .

Proof. By Lemma 4.4, the set

$$\mathcal{A}_0 = \{a \in F_\omega(X) : \text{supp}(a) \subset A, N_{f|\text{supp}(a)} \subset \dot{Y}\}$$

is dense in FX . Observe that for each $a \in \mathcal{A}_0$ we get $N_{f|\text{supp}(a)} \subset f(\text{supp}(a)) \cap \dot{Y} \subset f(\text{supp}(a)) \cap D_f \subset D_{f|\text{supp}(a)}$, which implies $N_{f|\text{supp}(a)} = \emptyset$ and $a \in \mathcal{A}_1$. Now we see that the density of the set \mathcal{A}_0 implies the density of the set $\mathcal{A}_1 \supset \mathcal{A}_0$ in FX . \square

5. 1-Mec functors and densely open squares

In this section we assume that $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ is a 1-mec functor and study its action on densely open squares. Let \mathcal{D} be a commutative square

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ p_X \downarrow & & \downarrow p_Y \\ X & \xrightarrow{f} & Y \end{array}$$

consisting of surjective maps between compact Hausdorff spaces. Applying the functor F to this square, we obtain the commutative square $F\mathcal{D}$:

$$\begin{array}{ccc} F\tilde{X} & \xrightarrow{F\tilde{f}} & F\tilde{Y} \\ Fp_X \downarrow & & \downarrow Fp_Y \\ FX & \xrightarrow{Ff} & FY. \end{array}$$

Lemma 5.1. *If the space X is first countable and the square \mathcal{D} is open at a non-empty subset $A \subset X$, then the square $F\mathcal{D}$ is open at the subset*

$$\mathcal{A}_1 = \{a \in F_\omega(X) : \text{supp}(a) \subset A, f|_{\text{supp}(a)} \text{ is 1-to-1}\} \subset FX.$$

If $\dot{Y} \subset D_f$ and the set A is dense in X , then the set \mathcal{A}_1 is dense in FX and hence the square $F\mathcal{D}$ is densely open.

Proof. Fix any point $b \in \mathcal{A}_1$ and consider its support $\text{supp}(b)$. If it is not empty, put $B = \text{supp}(b)$. If $\text{supp}(b)$ is empty, put $B = \{z\} \subset A$ be any singleton in A . In both cases we have that $B \subset A$, $f|_B$ is injective, and $b \in FB$, see Lemma 4.2. Let $C = f(B)$ and observe that $f|_B : B \rightarrow C$ is a homeomorphism. By Lemma 3.3, there is a closed subset $Z \subset X$ such that $B \subset D^{f|Z}$, $f(Z) = Y$ and $\tilde{f}(p_X^{-1}(Z)) = \tilde{Y}$.

Let $\tilde{Z} = p_X^{-1}(Z)$, $p_Z = p_X|_{\tilde{Z}}$, $f_Z = f|_Z$, $\tilde{f}_Z = \tilde{f}|_{\tilde{Z}}$ and consider the commutative square \mathcal{D}_Z :

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{\tilde{f}_Z} & \tilde{Y} \\ p_Z \downarrow & & \downarrow p_Y \\ Z & \xrightarrow{f_Z} & Y \end{array}$$

that consists of surjective maps. Applying to this square the epimorphic functor F , we obtain the commutative square $F\mathcal{D}_Z$:

$$\begin{array}{ccc} F\tilde{Z} & \xrightarrow{F\tilde{f}_Z} & F\tilde{Y} \\ Fp_Z \downarrow & & \downarrow Fp_Y \\ FZ & \xrightarrow{Ff_Z} & FY, \end{array}$$

also consisting of surjective maps.

Taking into account that $B \subset D^{f_Z}$ and F preserves finite 1-preimages, we conclude that $FB \subset D^{Ff_Z}$. By Lemma 3.1, the square $F\mathcal{D}_Z$ is open at FB . Applying Lemma 3.5, we conclude that the square $F\mathcal{D}$ is open at FB . In particular, $F\mathcal{D}$ is open at the point $b \in FB$.

If $\dot{Y} \subset D_f$, then by Lemma 4.5, the set \mathcal{A}_1 is dense in FX and hence the square $F\mathcal{D}$ is densely open. \square

6. 1-Mec functors and skeletal maps

In this section we study the action of 1-mec functors on some special types of skeletal maps. As in the preceding section, $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ is a 1-mec functor in the category of compact Hausdorff spaces. Our principal result is the following theorem.

Theorem 6.1. *For any surjective skeletal map $f : X \rightarrow Y$ between compact Hausdorff spaces the map $Ff : FX \rightarrow FY$ is skeletal at the subset*

$$\mathcal{A}_1 = \{a \in F_\omega(X) : f|_{\text{supp}(a)} \text{ is 1-to-1}\}.$$

If $\dot{Y} \subset D_f$, then the set \mathcal{A}_1 is dense in FX and hence the map Ff is skeletal.

Proof. By Theorem 2.7, the skeletal map $f : X \rightarrow Y$ can be identified with the limit map $\lim f_\alpha$ of a morphism $\vec{f} = \{f_\alpha\}_{\alpha \in \Sigma} : S_X \rightarrow S_Y$ between some ω -spectra $S_X = \{X_\alpha, p_\alpha^\beta, \Sigma\}$ and $S_Y = \{Y_\alpha, \pi_\alpha^\beta, \Sigma\}$ with surjective limit projections such that for any $\alpha \in \Sigma$ the limit square \mathcal{D}_α :

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ p_\alpha \downarrow & & \downarrow \pi_\alpha \\ X_\alpha & \xrightarrow{f_\alpha} & Y_\alpha \end{array}$$

is skeletal.

To show that the map Ff is skeletal at each point $a \in \mathcal{A}_1$, fix any open neighborhood $U \subset FX$ of a . We need to prove that the image $Ff(U)$ has non-empty interior in FY . The inclusion $a \in \mathcal{A}_1$ implies that the restriction $f|_{\text{supp}(a)}$ is injective.

By the continuity of the functor F , there is an index $\alpha \in \Sigma$ and an open neighborhood $U_\alpha \subset FX_\alpha$ of the point $a_\alpha = Fp_\alpha(a)$ such that $U \supset (Fp_\alpha)^{-1}(U_\alpha)$. Replacing α by a larger index, if necessary, we can additionally assume that the restriction $p_\alpha|_{\text{supp}(a)}$ and $\pi_\alpha \circ f|_{\text{supp}(a)}$ are injective. Then the map $f_\alpha|_{p_\alpha(\text{supp}(a))}$ also is injective. Since $\text{supp}(a_\alpha) \subset p_\alpha(\text{supp}(a))$, the restriction $f_\alpha|_{\text{supp}(a_\alpha)}$ is injective.

By our assumption the limit square \mathcal{D}_α is skeletal and by Lemma 2.6 it is open at some dense subset $A_\alpha \subset X_\alpha$. Repeating the argument from the proof of Lemma 4.4, we can approximate the element a_α by an element $a'_\alpha \in U_\alpha$ such that $\text{supp}(a'_\alpha) \subset A_\alpha$ and the map $f_\alpha|_{\text{supp}(a'_\alpha)}$ is injective. By Lemma 5.1, the square $F\mathcal{D}_\alpha$ is open at the point a'_α . Then for the neighborhood U_α of a'_α there is a non-empty open set $V_\alpha \subset FY_\alpha$ such that $V_\alpha \subset Ff_\alpha(U_\alpha)$ and the open subset $V = (F\pi_\alpha)^{-1}(V_\alpha)$ of the space FY lies in the image $Ff((Fp_\alpha)^{-1}(U_\alpha)) \subset Ff(U)$, which completes the proof of the skeletal-ity of Ff at a .

If $\dot{Y} \subset D_f$, then the set \mathcal{A}_1 is dense in FX by Lemma 4.5 and hence the map $Ff : FX \rightarrow FY$ is skeletal. \square

A map $f : X \rightarrow Y$ between topological spaces is called *irreducible* if $f(X) = Y$ but $f(Z) \neq Y$ for each closed subset $Z \subset X$.

Corollary 6.2. *For each irreducible map $f : X \rightarrow Y$ between compact Hausdorff spaces the map $Ff : FX \rightarrow FY$ is skeletal.*

Proof. This lemma follows from Theorem 6.1 because each closed irreducible map $f : X \rightarrow Y$ is skeletal and has $\dot{Y} \subset D_f$. \square

7. Preimage preserving functors and skeletal maps

The following theorem implies that for normal functors F the skeletality of a map $f : X \rightarrow Y$ between compacta follows from the skeletality of the map Ff .

Theorem 7.1. *Let $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ be a preimage preserving mec functor such that $F1 \neq F2$. A surjective map $f : X \rightarrow Y$ between compact Hausdorff spaces is skeletal if the map $Ff : FX \rightarrow FY$ is skeletal.*

Proof. Assume that the map Ff is skeletal. To show that $f : X \rightarrow Y$ is skeletal, fix a nowhere dense subset $N \subset Y$. We need to show that its preimage $f^{-1}(N)$ is nowhere dense in X . Assume conversely that $f^{-1}(N)$ contains some non-empty open set U . The set $F(X \setminus U)$ is closed in FX and its complement $\mathcal{U} = FX \setminus F(X \setminus U)$ is open in FX . Let us show that the set \mathcal{U} is not empty. Fix any point $u \in U$ and consider the closed subspace $Z = (X \setminus U) \cup \{u\}$ of X and the continuous map $p : Z \rightarrow 2 = \{0, 1\}$ such that $p^{-1}(0) = X \setminus U$ and $p^{-1}(1) = \{u\}$. By our hypothesis, $F1 \neq F2$. So we can find an element $b' \in F2 \setminus F1$. Since the functor F is epimorphic, there is an element $a' \in FZ$ such that $Fp(a') = b'$. The element a' does not belong to $F(X \setminus U)$, which implies that the sets $FZ \setminus F(X \setminus U)$ and $\mathcal{U} = FX \setminus F(X \setminus U)$ are not empty.

Since the map $Ff : FX \rightarrow FY$ is skeletal, the image $Ff(\mathcal{U})$ of the non-empty open set $\mathcal{U} \subset FX$ has non-empty interior in FY and hence contains some non-empty open subset $\mathcal{V} \subset Ff(\mathcal{U})$ of the space FY . Since the set N is nowhere dense in Y ,

the subspace $F_\omega(X \setminus N) = \{a \in F_\omega(X) : \text{supp}(a) \subset X \setminus N\}$ is dense in FX . So, we can find an element $b \in F_\omega(X) \cap \mathcal{V}$ with finite support $\text{supp}(b) \subset Y \setminus N$. Let $A = \text{supp}(b)$ if $\text{supp}(b) \neq \emptyset$ and $A = \{y\} \subset Y \setminus N$ be any singleton in $Y \setminus N$ if $\text{supp}(b) = \emptyset$. By [2], $b \in F(A) \subset F(X \setminus N)$. Since $b \in \mathcal{V} \subset Ff(\mathcal{U})$, there is an element $a \in \mathcal{U} = FX \setminus F(X \setminus U)$ with $Ff(a) = b$. Observe that $f^{-1}(A) \subset f^{-1}(Y \setminus N) = X \setminus f^{-1}(N) \subset X \setminus U$. Since the functor F preserves preimages, we conclude that $a \in F(f^{-1}(A)) \subset F(X \setminus U)$, which contradicts the choice of a . This contradiction shows that the set $f^{-1}(N)$ is nowhere dense in X and hence the map f is skeletal. \square

8. Proof of Theorem 1.1

To prove the “1-mec” part of Theorem 1.1, assume that $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ is a 1-mec functor such that for each compact zero-dimensional space Z the map $F2_Z : F(Z \oplus 2) \rightarrow F(Z \oplus 1)$ is skeletal.

Lemma 8.1. *For any surjective map $f : A \rightarrow B$ between finite discrete spaces and any compact zero-dimensional space Z the map $F(\text{id}_Z \oplus f) : F(Z \oplus A) \rightarrow F(Z \oplus B)$ is skeletal.*

Proof. Let $n = |A| \setminus |B|$ and $(A_i)_{i=0}^n$ be an increasing sequence of subsets of A such that $|A_0| = |B|$, $f(A_0) = B$, $A_n = A$ and $|A_{i+1} \setminus A_i| = 1$ for every $i < n$. For every positive number $i \leq n$ choose a surjective map $f_i : A_i \rightarrow A_{i-1}$ such that $f \circ f_i = f|_{A_i}$. Observe that

$$\text{id}_Z \oplus f = (\text{id}_Z \oplus f_1) \circ \cdots \circ (\text{id}_Z \oplus f_n)$$

and for every $i \leq n$ the map $\text{id}_Z \oplus f_i$ is homeomorphic to the map 2_{Z_i} where $Z_i = Z \oplus (A_{i-1} \setminus D_{f_i})$. By our assumption the map $F(2_{Z_i})$ is skeletal and so is its homeomorphic copy $F(\text{id}_Z \oplus f_i)$. Since the composition of skeletal maps between compacta is skeletal, the map

$$F(\text{id}_Z \oplus f) = F(\text{id}_Z \oplus f_1) \circ \cdots \circ F(\text{id}_Z \oplus f_n)$$

is skeletal. \square

Lemma 8.2. *For any surjective map $f : A \rightarrow B$ between finite discrete spaces and any compact space X the map $F(\text{id}_X \oplus f) : F(X \oplus A) \rightarrow F(X \oplus B)$ is skeletal.*

Proof. By [5, 3.2.2, 3.1.C], the compact space X is the image of a compact zero-dimensional space Z under an irreducible map $\xi : Z \rightarrow X$. Applying to the commutative diagram

$$\begin{array}{ccc} X \oplus A & \xrightarrow{\text{id}_X \oplus f} & X \oplus B \\ \uparrow \xi \oplus \text{id}_A & & \uparrow \xi \oplus \text{id}_B \\ Z \oplus A & \xrightarrow{\text{id}_Z \oplus f} & Z \oplus B \end{array}$$

the functor F , we obtain the commutative diagram

$$\begin{array}{ccc} F(X \oplus A) & \xrightarrow{F(\text{id}_X \oplus f)} & F(X \oplus B) \\ \uparrow F(\xi \oplus \text{id}_A) & & \uparrow F(\xi \oplus \text{id}_B) \\ F(Z \oplus A) & \xrightarrow{F(\text{id}_Z \oplus f)} & F(Z \oplus B) \end{array}$$

in which the map $F(\xi \oplus \text{id}_A)$ is surjective, $F(\text{id}_Z \oplus f)$ is skeletal by Lemma 8.1 and $F(\xi \oplus \text{id}_B)$ is skeletal by Corollary 6.2. Because of that the map $F(\text{id}_X \oplus f)$ is skeletal. \square

The following lemma yields the “1-mec” part of Theorem 1.1.

Lemma 8.3. *For any skeletal surjection $f : X \rightarrow Y$ between compacta the map $Ff : FX \rightarrow FY$ is skeletal.*

Proof. By Lemma 4.4, the set

$$\mathcal{A}_0 = \{a \in F_\omega(X) : N_{f|\text{supp}(a)} \subset \dot{Y}\}$$

is dense in FX . So, the skeletality of the map Ff will follow as soon as we check its skeletality at each point $a \in \mathcal{A}_0$. If $f|_{\text{supp}(a)}$ is injective, then Ff is skeletal at a by Theorem 6.1. So, we assume that $f|_{\text{supp}(a)}$ is not injective. In this case the support $A = \text{supp}(a)$ is not empty and $a \in FA$ by Lemma 4.2. By our assumption, $N_{f|A} \subset \dot{Y}$ and hence the complement

$Y \setminus N_{f|A}$ is an open-and-closed subset of Y . Consider the closed subspace $\tilde{X} = f^{-1}(Y \setminus N_{f|A}) \cup N^{f|A}$ of X and the topological sum $\tilde{Y} = (Y \setminus N_{f|A}) \oplus N^{f|A}$. Next, consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i \uparrow & & \uparrow h \\ \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \end{array}$$

where $i: \tilde{X} \rightarrow X$ is the embedding, \tilde{f} is defined by $\tilde{f}|_{\tilde{X} \setminus N^{f|A}} = f|_{\tilde{X} \setminus N^{f|A}}$ and $\tilde{f}|_{N^{f|A}} = \text{id}$ while $h: \tilde{Y} \rightarrow Y$ is defined by $h|_{Y \setminus N_{f|A}} = \text{id}$ and $h|_{N^{f|A}} = f|_{N^{f|A}}$.

Applying the functor F to this diagram we get the commutative diagram

$$\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ Fi \uparrow & & \uparrow Fh \\ F\tilde{X} & \xrightarrow{F\tilde{f}} & F\tilde{Y} \end{array}$$

Since \tilde{f} is skeletal and the restriction $\tilde{f}|_A$ is injective, the map $F\tilde{f}: F\tilde{X} \rightarrow F\tilde{Y}$ is skeletal at a by Theorem 6.1. By Lemma 8.2, the map Fh is skeletal. Consequently, the composition $Fh \circ F\tilde{f}$ is skeletal at a and then Ff is skeletal at a by Lemma 3.4. \square

To prove the “1-mecw” part of Theorem 1.1, assume that $F: \mathbf{Comp} \rightarrow \mathbf{Comp}$ is a 1-mecw functor such that for each zero-dimensional compact metrizable space Z the map $F2_Z: F(Z \oplus 2) \rightarrow F(Z \oplus 1)$ is skeletal. The skeletality of the functor F will follow from the “1-mec” part of Theorem 1.1 as soon as we check that for each zero-dimensional compact space Z the map $F2_Z: F(Z \oplus 2) \rightarrow F(Z \oplus 1)$ is skeletal. For this we shall apply the characterization Theorem 2.7.

By (the proof of) Proposition 1.3.5 of [3], the zero-dimensional space Z is homeomorphic to the limit $\lim \mathcal{S}_Z$ of an ω -spectrum $\mathcal{S}_Z = \{Z_\alpha, p_\alpha^\beta, \Sigma\}$ with surjective limit projections, consisting of zero-dimensional compact metrizable spaces Z_α , $\alpha \in \Sigma$. For $n \in \{1, 2\}$, consider the inverse spectrum $\mathcal{S}_Z \oplus n = \{Z_\alpha \oplus n, p_\alpha^\beta \oplus \text{id}_n, \Sigma\}$, where $\text{id}_n: n \rightarrow n$ denotes the identity map of the discrete space $n = \{0, \dots, n-1\}$. Next, consider the skeletal morphism $\{2_{Z_\alpha}\}_{\alpha \in \Sigma}: \mathcal{S}_Z \oplus 2 \rightarrow \mathcal{S}_Z \oplus 1$. Applying to this morphism the mecw functor F , we obtain a morphism $\{F2_{Z_\alpha}\}_{\alpha \in \Sigma}: F(\mathcal{S}_Z \oplus 2) \rightarrow F(\mathcal{S}_Z \oplus 1)$. By our assumption, for every $\alpha \in \Sigma$ the map $F2_{Z_\alpha}: F(Z_\alpha \oplus 2) \rightarrow F(Z_\alpha \oplus 1)$ is skeletal. Then Theorem 2.7 guarantees that the limit map $\lim F2_{Z_\alpha}: \lim F(\mathcal{S}_Z \oplus 2) \rightarrow \lim F(\mathcal{S}_Z \oplus 1)$ of the skeletal morphism $\{F2_{Z_\alpha}\}_{\alpha \in \Sigma}$ is skeletal. By the continuity of the functor F , this map is homeomorphic to the map $F2_Z: F(Z \oplus 2) \rightarrow F(Z \oplus 1)$.

9. Proof of Theorem 1.3

We need to prove that a 1-mecw functor $F: \mathbf{Comp} \rightarrow \mathbf{Comp}$ is skeletal if it is finitely bicommutative and finitely skeletal. By Theorem 1.1, it suffices to check that for any zero-dimensional compact metrizable space Z the map $F2_Z: F(Z \oplus 2) \rightarrow F(Z \oplus 1)$ is skeletal. Write the space Z as the limit of an inverse spectrum $\mathcal{S}_Z = \{Z_n, p_n^m, \omega\}$ consisting of finite spaces Z_n , $n \in \omega$, and surjective bonding maps $p_n^m: Z_m \rightarrow Z_n$, $n \leq m$. Then the map $2_Z: Z \oplus 2 \rightarrow Z \oplus 1$ can be identified with the limit map of the morphism $\{2_{Z_n}\}_{n \in \omega}: \mathcal{S}_Z \oplus 2 \rightarrow \mathcal{S}_Z \oplus 1$ between the inverse spectra $\mathcal{S}_Z \oplus 2 = \{Z_n \oplus 2, p_n^m \oplus \text{id}_2, \omega\}$ and $\mathcal{S}_Z \oplus 1 = \{Z_n \oplus 1, p_n^m \oplus \text{id}_1, \omega\}$. Applying to this morphism the continuous functor F , we obtain a morphism $\{F2_{Z_n}\}_{n \in \omega}: F(\mathcal{S}_Z \oplus 2) \rightarrow F(\mathcal{S}_Z \oplus 1)$ between the inverse spectra $F(\mathcal{S}_Z \oplus 2) = \{F(Z_n \oplus 2), F(p_n^m \oplus \text{id}_2), \omega\}$ and $F(\mathcal{S}_Z \oplus 1) = \{F(Z_n \oplus 1), F(p_n^m \oplus \text{id}_1), \omega\}$.

The finite skeletality of the functor F implies that the morphism $\{F2_{Z_n}\}_{n \in \omega}$ consists of skeletal maps $F2_{Z_n}: F(Z_n \oplus 2) \rightarrow F(Z_n \oplus 1)$ for all $n \in \omega$.

It is clear that for any $n \leq m$ the bonding \downarrow_n^m -square \mathcal{D}_n^m

$$\begin{array}{ccc} Z_m \oplus 2 & \xrightarrow{2_{Z_m}} & Z_m \oplus 1 \\ p_n^m \oplus \text{id}_2 \downarrow & & \downarrow p_n^m \oplus \text{id}_1 \\ Z_n \oplus 2 & \xrightarrow{2_{Z_n}} & Z_n \oplus 1 \end{array}$$

is bicommutative and consists of finite spaces. Since the functor F is finitely bicommutative, the bonding \downarrow_n^m -square $F\mathcal{D}_n^m$ of the morphism $\{F2_{Z_n}\}_{n \in \omega}$ also is bicommutative.

By Proposition 2.5 of [11], the bicommutativity of the bonding \downarrow_n^m -squares $F\mathcal{D}_n^m$, $n \leq m$, implies the bicommutativity of the limit \downarrow_n -square $F\mathcal{D}_n$

$$\begin{array}{ccc}
 F(Z \oplus 2) & \xrightarrow{F(2_Z)} & F(Z \oplus 1) \\
 \downarrow F(p_n \oplus \text{id}_2) & & \downarrow F(p_n \oplus \text{id}_1) \\
 F(Z_n \oplus 2) & \xrightarrow{F(2_{Z_n})} & F(Z_n \oplus 1)
 \end{array}$$

for every $n \in \omega$. This fact combined with the skeletality of the map $F(2_{Z_n})$ implies that the limit \downarrow_n -square $F\mathcal{D}_n$ is skeletal. Now Proposition 3.1 of [1] guarantees that the limit map $F2_Z : F(Z \oplus 2) \rightarrow F(Z \oplus 1)$ of the morphism $\{F2_{Z_n}\}_{n \in \omega} : F(Z_n \oplus 2) \rightarrow F(Z_n \oplus 1)$ is skeletal.

10. Proof of Theorem 1.5

Let $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ be a 1-mecw functor. Given a skeletally generated compact Hausdorff space X , we need to prove that the space FX is skeletally generated.

Represent X as the limit of a continuous ω -spectrum $\mathcal{S} = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ with surjective limit projections $p_\alpha : X \rightarrow X_\alpha$, $\alpha \in \Sigma$. By [8], the space X , being skeletally generated, has countable cellularity. Consequently, the set \dot{X} of isolated points of X is at most countable.

For each isolated point $x \in \dot{X}$ of X we can find an index $\alpha_x \in \Sigma$ such that $\{x\} = \pi_{\alpha_x}^{-1}(U_x)$ for some open set $U_x \subset X_{\alpha_x}$, which must coincide with the singleton of $\pi_{\alpha_x}(x)$. Then for any $\alpha \geq \sup\{\alpha_x : x \in \dot{X}\}$ we get $\pi_\alpha(\dot{X}) \subset D_{\pi_\alpha} \cap \dot{X}_\alpha$. Replacing the index set Σ by its cofinal subset $\{\alpha \in \Sigma : \alpha \geq \sup\{\alpha_x : x \in \dot{X}\}\}$, if necessary, we can assume that $\pi_\alpha(\dot{X}) \subset D_{\pi_\alpha} \cap \dot{X}_\alpha$ for all $\alpha \in \Sigma$.

Claim 10.1. *The set $\Sigma' = \{\alpha \in \Sigma : \pi_\alpha(\dot{X}) = \dot{X}_\alpha\}$ is closed and cofinal in Σ .*

Proof. First we prove that Σ' is closed in Σ . Given a chain $C \subset \Sigma'$ having the supremum $\sup C$ in Σ , we need to show that $\sup C \in \Sigma'$. If $\sup C \in C$, then there is nothing to prove. So we assume that $\gamma = \sup C \notin C$. In this case by the continuity of the spectrum \mathcal{S} , the space X_γ is the limit of the inverse subspectrum $\mathcal{S}|_C = \{X_\alpha, \pi_\alpha^\beta, C\}$.

We need to prove that $\dot{X}_\gamma \subset \pi_\gamma(\dot{X})$. Take any isolated point $x \in \dot{X}_\gamma$. By the definition of the topology of the inverse limit $X_\gamma = \lim \mathcal{S}|_C$, there is an index $\alpha \in C$ such that $\{x\} = (\pi_\alpha^\gamma)^{-1}(y)$ for some isolated point $y \in X_\alpha$. Since $y \in \dot{X}_\alpha = \pi_\alpha(\dot{X})$, there is a point $z \in \dot{X}$ with $y = \pi_\alpha(z)$. Now consider the point $x' = \pi_\gamma(z) \in X_\gamma$ and observe that $\pi_\alpha^\gamma(x') = \pi_\alpha^\gamma(\pi_\gamma(z)) = \pi_\gamma(z) = y = \pi_\alpha^\gamma(x)$ and $y \in \dot{X}_\alpha = \pi_\alpha(\dot{X}) \subset D_{\pi_\alpha} \subset D_{\pi_\alpha^\gamma}$, which implies $x = x' \in \pi_\gamma(\dot{X})$.

Next, we prove that Σ' is cofinal in Σ . Given any $\alpha_0 \in \Sigma$ we need to find $\alpha \in \Sigma'$ with $\alpha \geq \alpha_0$. For any isolated point $x \in \dot{X}_{\alpha_0} \setminus \pi_{\alpha_0}(\dot{X})$ the preimage $\pi_{\alpha_0}^{-1}(x)$ is an open subset of X containing no isolated points of X . Since the metrizable compactum X_{α_0} has at most countably many isolated points, and the index set Σ is ω -complete, there is an index $\alpha_1 \geq \alpha_0$ such that for each isolated point $x \in \dot{X}_{\alpha_0} \setminus \pi_{\alpha_0}(\dot{X})$ the preimage $(\pi_{\alpha_0}^{\alpha_1})^{-1}(x)$ is not a singleton, which means that $\dot{X}_{\alpha_0} \setminus \pi_{\alpha_0}(\dot{X}) \subset N_{\pi_{\alpha_0}}^{\alpha_1}$. Proceeding by induction, we can construct an increasing chain $(\alpha_n)_{n \in \omega}$ in Σ such that $\dot{X}_{\alpha_n} \setminus \pi_{\alpha_n}(\dot{X}) \subset N_{\pi_{\alpha_n}}^{\alpha_{n+1}}$ for all $n \in \omega$. Since Σ is ω -complete, the chain $(\alpha_n)_{n \in \omega}$ has the smallest upper bound $\alpha_\omega = \sup_{n \in \omega} \alpha_n$ in Σ . We claim that $\alpha_\omega \in \Sigma'$. Given any isolated point $x \in \dot{X}_{\alpha_\omega}$, we need to prove that $x \in \pi_{\alpha_\omega}(\dot{X})$.

The continuity of the spectrum \mathcal{S} guarantees that the space X_{α_ω} is the limits of the inverse sequence $\{X_{\alpha_n}, \pi_{\alpha_n}^{\alpha_{n+1}}, \omega\}$. By the definition of the topology of the inverse limit, there is a number $n \in \omega$ such that $\{x\} = (\pi_{\alpha_n}^{\alpha_\omega})^{-1}(U_n)$ for some open set $U_n \subset X_{\alpha_n}$ which must coincide with the singleton of the isolated point $y = \pi_{\alpha_n}^{\alpha_\omega}(x)$. We claim that $y \in \pi_{\alpha_n}(\dot{X})$. In the other case the choice of the index α_{n+1} guarantees that the preimage $(\pi_{\alpha_n}^{\alpha_{n+1}})^{-1}(y)$ is not a singleton and then

$$(\pi_{\alpha_{n+1}}^{\alpha_\omega})^{-1}((\pi_{\alpha_n}^{\alpha_{n+1}})^{-1}(y)) = (\pi_{\alpha_n}^{\alpha_\omega})^{-1}(y) = \{x\}$$

is not a singleton, which is a contradiction. Thus $y = \pi_{\alpha_n}(z)$ for some isolated point $z \in \dot{X}$. Taking into account that $y \in \pi_{\alpha_n}(\dot{X}) \subset D_{\pi_{\alpha_n}} \subset D_{\pi_{\alpha_n}^{\alpha_\omega}}$ and $\pi_{\alpha_n}^{\alpha_\omega}(x) = y = \pi_{\alpha_n}^{\alpha_\omega}(\pi_{\alpha_n}(z))$, we can show that $x = \pi_{\alpha_\omega}(z) \in \pi_{\alpha_\omega}(\dot{X})$. \square

It follows from Claim 10.1 that X is the limit of the ω -spectrum $\mathcal{S} = \{X_\alpha, \pi_\alpha^\beta, \Sigma'\}$ consisting of metrizable compacta and surjective skeletal bonding projections and such that $\dot{X}_\alpha \subset D_{\pi_\alpha} \subset D_{\pi_\alpha^\beta}$ for all $\alpha \leq \beta$ in Σ' . By Theorem 6.1, the latter condition guarantees that the map $F\pi_\alpha^\beta : FX_\beta \rightarrow FX_\alpha$ is skeletal. Since the functor F is epimorphic, continuous and preserves weight, the space FX is skeletally generated, being the limit of the continuous ω -spectrum $\{FX_\alpha, F\pi_\alpha^\beta, \Sigma'\}$ with surjective skeletal bonding projections $F\pi_\alpha^\beta : FX_\beta \rightarrow FX_\alpha$.

11. Proof of Theorem 1.6

Assume that $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ is a preimage preserving mecw functor with $F1 \neq F2$. We need to prove that a compact Hausdorff space X is skeletally generated if and only if so is the space FX .

If X is skeletally generated, then by Theorem 1.5, so is the space FX . Now assume conversely that the space FX is skeletally generated.

Write the space X as the limit of an inverse ω -spectrum $\mathcal{S} = \{X_\alpha, p_\alpha^\beta, \Sigma\}$ with surjective bonding projections. Applying to this spectrum the functor F , we get the inverse ω -spectrum $F\mathcal{S} = \{FX_\alpha, Fp_\alpha^\beta, \Sigma\}$. By the continuity of the functor F , the limit space of the spectrum $F\mathcal{S}$ can be identified with FX . The space FX , being skeletally generated, is the limit of an inverse ω -spectrum with skeletal bonding projection. By the Spectral Theorem of Ščepin [11], [3, 1.3.4], we can assume that the latter spectrum coincides with the subspectrum $\mathcal{S}|\Sigma' = \{FX_\alpha, Fp_\alpha^\beta, \Sigma'\}$ for some ω -closed cofinal subset Σ' of the index set Σ . According to Theorem 7.1, the skeletality of the maps Fp_α^β implies the skeletality of the maps p_α^β for any $\alpha \leq \beta$ in Σ' . Consequently, the space X is skeletally generated, being homeomorphic to the limit space of the inverse spectrum $\mathcal{S}|\Sigma' = \{X_\alpha, p_\alpha^\beta, \Sigma'\}$ with surjective skeletal bonding projections.

12. Some examples and open problems

In this section we shall present examples of skeletal and non-skeletal functors.

For a natural number n and a mec functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ let F_n be the subfunctor of F assigning to each compact space X the closed subspace

$$F_n(X) = \{a \in FX : \exists \xi \in C(n, X) \text{ such that } a \in F\xi(Fn)\}$$

of F . First we observe that subfunctors F_n of open functors need not be skeletal.

Example 12.1. For the open functor $P : \mathbf{Comp} \rightarrow \mathbf{Comp}$ of probability measures and every natural number $n \geq 2$ the subfunctor P_n is not skeletal.

This can be shown applying Theorem 1.1. The non-skeletal functors P_n are not finitary. We recall that a functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ is *finitary* if for any finite discrete space X the space FX is finite. A typical example of a finitary functor is the functor \exp of hyperspace, see [12, 2.1.1]. This functor is open according to [12, 2.10.11].

Example 12.2. For every $n \geq 3$ the subfunctor \exp_n of the hyperspace functor is normal and finitary but not skeletal.

By Corollary 1.2, each open 1-mec functor is skeletal. Now we present three examples showing that the reverse implication does not hold.

By Proposition 2.10.1 [12], a normal functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ with finite supports is bicommutative if and only if F is finitely bicommutative. In [12, p. 85] A. Teleiko and M. Zarichnyi constructed an example of a finitary normal functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$, which is finitely bicommutative but not bicommutative. Applying to this functor Theorems 1.3 and 1.4, we get:

Example 12.3. There is a finitary normal functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ which is finitely bicommutative and skeletal but is not bicommutative and hence not open.

By Proposition 2.10.1 of [12], the functor from Example 12.3 has infinite degree. There is also a finitary *weakly normal* functor of finite degree, which is skeletal but not open.

By $\lambda : \mathbf{Comp} \rightarrow \mathbf{Comp}$ we denote the functor of superextension, see [12, 2.1.2]. It is known that the functor λ is open, finitary, weakly normal, preserves 1-preimages but fails to preserve preimages, see [7] and Propositions 2.3.2, 2.10.13 of [12]. By [12, 2.10.19], for every $n \geq 3$ the subfunctor λ_3 of λ is not open. Using the characterization Theorem 1.1, one can easily check that the functor λ_3 is skeletal. Thus we obtain another:

Example 12.4. The finitary weakly normal functor λ_3 is skeletal but not open.

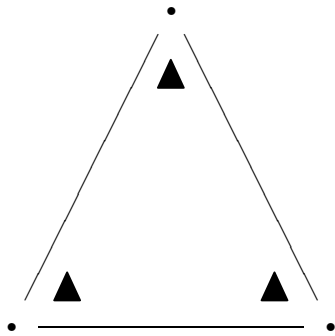
The functor λ_3 is finitary and has finite degree but is not normal. Our final example is a normal functor of finite degree which is skeletal but not open.

Example 12.5. The functor P_3 contains a normal subfunctor P_Δ , which is skeletal but not open.

Proof. In the standard 2-simplex $\Delta_2 = \{(\alpha, \beta, \gamma) \in [0, 1]^3 : \alpha + \beta + \gamma = 1\}$ consider the closed subsets

$$\begin{aligned} \Delta_0 &= \{(\alpha, \beta, \gamma) \in \Delta^2 : \max\{\alpha, \beta, \gamma\} = 1\}, \\ \Delta_1 &= \left\{(\alpha, \beta, \gamma) \in \Delta^2 : \min\{\alpha, \beta, \gamma\} = 0, \max\{\alpha, \beta, \gamma\} \leq \frac{11}{12}\right\}, \\ \Delta_2 &= \left\{(\alpha, \beta, \gamma) \in \Delta^2 : \min\{\alpha, \beta, \gamma\} \geq \frac{1}{12}, \max\{\alpha, \beta, \gamma\} \geq \frac{3}{4}\right\}, \end{aligned}$$

and their union $\Delta = \Delta_0 \cup \Delta_1 \cup \Delta_2$, which looks as follows:



Now consider the subfunctor $P_\Delta \subset P$ of the functor of probability measures assigning to each compact space X the closed subspace

$$P_\Delta(X) = \{\alpha\delta_x + \beta\delta_y + \gamma\delta_z : (\alpha, \beta, \gamma) \in \Delta, x, y, z \in X\} \subset P(X).$$

Here δ_x stands for the Dirac measures concentrated at a point x . One can check that P_Δ is a normal functor of degree $\deg P_\Delta = 3$. In fact, P_Δ is a subfunctor of the functor $P_3 \subset P$. Theorem 2.10.21 [12] characterizing open normal functors of finite degree implies that the functor P_Δ is not open. Applying the characterization Theorem 1.1, one can check that the functor P_Δ is skeletal. \square

Examples 12.1–12.5 suggest the following open

Problem 12.6. Assume that a finitary normal functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ of finite degree is skeletal. Is F open? Equivalently, is F finitely bicommutative?

Let us also ask some other questions about skeletality of functors.

We shall say that a functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ is (finitely) *square-skeletal* if for each skeletal square \mathcal{D} consisting of continuous surjective maps between (finite) compact spaces the square $F\mathcal{D}$ is skeletal.

Proposition 12.7. Let $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ be an epimorphic functor.

- (1) If F is (finitely) square-skeletal, then F is (finitely) skeletal.
- (2) If F is (finitely) bicommutative and (finitely) skeletal, then F is (finitely) square-skeletal.
- (3) If F is finitary, then F is finitely square-skeletal if and only if F is finitely bicommutative only if F is skeletal.

Proof. (1), (2) The first two statements follow from Remarks 2.5 and 2.4, respectively.

(3) The third statement follows from Theorem 1.3 and an observation that a commutative square consisting of epimorphisms between finite spaces is skeletal if and only if it is bicommutative. \square

Proposition 12.7 suggests another two problems:

Problem 12.8. Is each (finitary) skeletal normal functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ (finitely) square-skeletal?

Problem 12.9. Is a normal functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ skeletal if it is finitely square-skeletal? (Theorem 1.3 implies that the answer is affirmative if the functor F is finitary).

It is clear that each functor that preserves (1-)preimages preserves finite (1-)preimages. We do not know if the converse statement is true.

Problem 12.10. Does a mec functor $F : \mathbf{Comp} \rightarrow \mathbf{Comp}$ preserve (1-)preimages if F preserves finite (1-)preimages.

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